STA5002: Mathematical Statistics

Assignment 2 Solution (Oct 18th – Oct 27th)

Note: The solutions only serve as a reference. Some problems may have different methods to reach the same answer.

1. Suppose the continuous random variables X and Y have the joint PDF as (10 points)

$$f(x,y) = \frac{3}{2}$$

for $x^2 \leq y \leq 1,\ 0 < x < 1$. What is the conditional distribution of Y given $X = x$.

Solution: The marginal distribution of x is

$$f_x(x) = \int_{x^2}^{1} \frac{3}{2} dy = \frac{3}{2}(1 - x^2), 0 < x < 1,$$

and thus the conditional distribution is

$$f_{y|x}(y|x) = \frac{f(x,y)}{f_x(x)} = \frac{\frac{3}{2}}{\frac{3}{2}(1 - x^2)} = \frac{1}{1 - x^2}, x^2 \leq y \leq 1.$$

2. Suppose that the joint PDF of (X, Y) is

$$f(x,y) = c\sqrt{1 - x^2 - y^2}, x^2 + y^2 \leq 1.$$

Find the marginal PDF $f_X(x)$ and the constant c. (Hint: consider transformations like $y = a\sin(\theta)$ when calculating the integral.) (10 points)

Solution: The marginal PDF $f_X(x)$ is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = c\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}dy.$$

For fixed x, let $y = \sqrt{1 - x^2}\sin(\theta)$, and then $dy = \sqrt{1 - x^2}\cos(\theta)\, d\theta$. Thus

$$f_X(x) = c\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1 - x^2 - y^2}dy = c(1 - x^2)\int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta.$$

As $\cos^2(\theta) = (\cos(2\theta) + 1)/2$, it is not difficult to obtain

$$\int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2} [\cos(2\theta) + 1] d\theta = \frac{\pi}{2}.$$

Therefore,

$$f_X(x) = \frac{c\pi}{2}(1 - x^2), -1 \le x \le 1.$$

Since the PDF should integrate to 1, we have

$$1 = \int_{-\infty}^{\infty} f_X(x)dx = \frac{c\pi}{2}\int_{-1}^{1}(1 - x^2)dx = \frac{2c\pi}{3} \Longrightarrow c = \frac{3}{2\pi}.$$

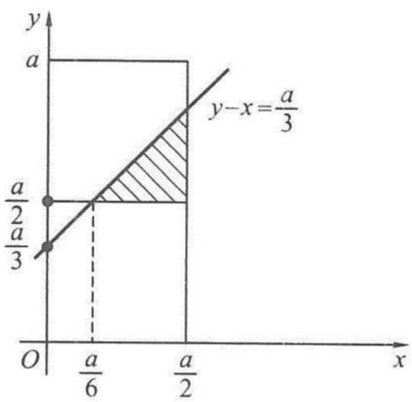
$$\Longrightarrow f_X(x) = \frac{3}{4}(1 - x^2), -1 \le x \le 1.$$

3. For a stick with length a, randomly pick one point on the left-hand side and right-hand side of the middle point of the stick, respectively. Compute the probability that the distance between the two picked points is less than $\frac{a}{3}$. (10 points)

Solution: Let X denote the distance between the point picked on the left-hand side of the middle point and the left endpoint of the stick, and then $X \sim U(0, a/2)$. Let Y denote the distance between the point picked on the right-hand side of the middle point and the left endpoint of the stick, and then $Y \sim U(a/2, a)$. Since X and Y are independent, the joint PDF of (X, Y) is

$$f(x, y) = \begin{cases} \frac{4}{a^2}, 0 < x < \frac{a}{2}, \frac{a}{2} < y < a \\ 0, \quad \text{otherwise} \end{cases}.$$

The probability to be computed is $P(Y - X < a/3)$. The integration area is shown below



Therefore

$$P\left(Y-X<\frac{a}{3}\right)=\int_{a/6}^{a/2}\int_{a/2}^{a/3+x}\frac{4}{a^2}dydx=\frac{2}{9}.$$

4. The joint PDF of the random variables X_1 and X_2 is

$$f_{X_1,X_2}(x_1,x_2)=k\exp\left\{-\left(\frac{x_1^2}{6}-\frac{x_1x_2}{3}+\frac{2x_2^2}{3}\right)\right\},-\infty<x_1,x_2<\infty.$$

Find k, $E(X_1)$, $E(X_2)$, $\mathrm{Var}(X_1)$, $\mathrm{Var}(X_2)$, $\mathrm{Cov}(X_1,X_2)$ and $\mathrm{Corr}(X_1,X_2)$. (10 points)

Solution: By looking at the form of $f_{X_1,X_2}(x_1,x_2)$, we find that it resembles the PDF of a bivariate normal distribution:

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}.$$

Comparing the two PDFs, we have:

$$\mu_1=0,\mu_2=0,2(1-\rho^2)\sigma_1^2=6,2(1-\rho^2)\sigma_2^2=\frac{3}{2},\frac{\sigma_1\sigma_2(1-\rho^2)}{\rho}=3.$$

Solving the equations, we obtain

$$\rho=\frac{1}{2},\sigma_1^2=4,\sigma_2^2=1\Longrightarrow k=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}=\frac{1}{2\sqrt{3}\pi},$$

and $E(X_1)=\mu_1=0$, $E(X_2)=\mu_2=0$, $\mathrm{Var}(X_1)=\sigma_1^2=4$, $\mathrm{Var}(X_2)=\sigma_2^2=1$, $\mathrm{Cov}(X_1,X_2)=\rho\sigma_1\sigma_2=1$, $\mathrm{Corr}(X_1,X_2)=1/2$.

5. Suppose that (X,Y) follows a bivariate normal distribution $N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$ with $\mu_1=\mu_2=0$, $\sigma_1^2=1$, $\sigma_2^2=2$, and $\rho=\frac{1}{\sqrt{2}}$. Let $U=aX+bY$, $V=cX+dY$, find the four equations that constants a,b,c,d need to satisfy such that U and V are independent standard normal random variables (no need to solve the equations). (10 points)

Solution: Based on the description, the joint PDF of (X,Y) is

$$f_{X,Y}(x,y)=\frac{1}{2\pi}\exp\left\{-\left[x^2-xy+\frac{y^2}{2}\right]\right\}.$$

Consider the joint PDF of (U,V), and the Jacobian is

$$J = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^{-1} = \frac{1}{ad - bc}.$$

The inverse transformations are $X = (dU - bV)/(ad - bc)$, $Y = (aV - cU)/(ad - bc)$, and thus the joint PDF of (U, V) is

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{du - bv}{ad - bc}, \frac{av - cu}{ad - bc}\right)|J|$$

$$= \frac{1}{2\pi|ad - bc|}\exp\left\{-\frac{1}{(ad - bc)^2}\left[(du - bv)^2 - (du - bv)(av - cu) + \frac{(av - cu)^2}{2}\right]\right\}$$

$$= \frac{1}{2\pi|ad - bc|}\exp\left\{-\frac{1}{(ad - bc)^2}\left[\left(d^2 + cd + \frac{c^2}{2}\right)u^2 - (2bd + ad + bc + ac)uv\right.\right.$$

$$\left.\left. + \left(b^2 + ab + \frac{a^2}{2}\right)v^2\right]\right\}.$$

For U and V to be independent standard normal random variables, we have

$$f_{U,V}(u,v) = f_U(u)f_V(v) = \frac{1}{2\pi}\exp\left\{-\frac{(u^2 + v^2)}{2}\right\}.$$

For the two joint PDFs to be equal, constants a, b, c, d need to satisfy the following four equations:

$$|ad - bc| = 1, d^2 + cd + \frac{c^2}{2} = \frac{1}{2}, b^2 + ab + \frac{a^2}{2} = \frac{1}{2}, 2bd + ad + bc + ac = 0.$$

6. Suppose that we generate a random point (X, Y) inside the unit disk as follows: we generate the radius R from $U[0,1]$, and generate the angle Θ independent of R from $U[0, 2\pi]$. Then we let $X = R\cos\Theta$ and $Y = R\sin\Theta$. Find the joint PDF of (X, Y) and the marginal PDFs of X and Y respectively. (10 points)

Solution:

Since R and Θ are independent, the joint distribution of R and Θ is

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi}, 0 \le r \le 1, 0 \le \theta \le 2\pi,$$

As $\begin{cases} x = R\cos\ \Theta \\ y = R\sin\ \Theta \end{cases}$, we have

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\mathrm{r}\sin\theta \\ \sin\theta & \mathrm{r}\cos\theta \end{vmatrix} = \mathrm{r} = \sqrt{x^2 + y^2}.$$

Then the joint distribution of X and Y is

$$f_{X,Y}(x,y) = \frac{1}{2\pi}|J|^{-1} = \frac{1}{2\pi}\frac{1}{\sqrt{x^2+y^2}},\ x^2+y^2 \le 1.$$

The marginal distribution of X is

$$f_x(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2\pi\sqrt{x^2+y^2}}dy$$

$$= \int_0^{\sqrt{1-x^2}} \frac{1}{\pi\sqrt{x^2+y^2}}dy$$

$$= \frac{1}{2\pi}\log\frac{(\sqrt{1-x^2}+1)^2}{x^2}, -1 \le x \le 1.$$

Symmetrically,

$$f_y(y) = \frac{1}{2\pi}\log\frac{(\sqrt{1-y^2}+1)^2}{y^2}, -1 \le y \le 1.$$

7. Assume that $X \sim Exp(1)$ and $Y \sim Exp(1)$ are independent. Are $U = X + Y$ and $V = \frac{X}{X+Y}$ independent? Justify your answer. (10 points)

Solution: Since $X \sim Exp(1)$ and $Y \sim Exp(1)$ are independent, the joint PDF of (X, Y) is

$$f_{X,Y}(x,y) = \exp\{-(x+y)\}, x > 0, y > 0.$$

$$\begin{cases} u = x + y \\ v = x/(x+y) \end{cases} \Rightarrow \begin{cases} x = uv \\ y = u(1-v) \end{cases} \Rightarrow J = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -uv - u(1-v) = -u\,.$$

Moreover, $u > 0$, $0 < v < 1$. Therefore, the joint PDF of (U, V) is

$$f_{U,V}(u,v) = f_{X,Y}(x,y)|J| = \exp\{-(x+y)\}u = ue^{-u}, u > 0.$$

Since $f_{U,V}(u,v)$ can be decomposed as $g(u)h(v)$, where $g(u) = ue^{-u}$ and $h(v) = 1$, U and V are independent.

8. Suppose that an equipment consists of three electronic components of the same type and the equipment works only if the three components are all working. It is known that the three components work independently, and their lifetime all follow $Exp(\lambda)$. Let T be the equipment's lifetime. Find the PDF of T and compute $E(T)$. (10 points)

Solution: Let T_i denote the lifetime of the ith component, $i = 1, 2, 3$. Then T_1, T_2, T_3 are independent and the CDF and PDF for each component are ($i = 1, 2, 3$)

$$f(t) = \lambda e^{-\lambda t}, F(t) = 1 - e^{-\lambda t}, t > 0.$$

By the description in the problem, we know that $T = \min\{T_1, T_2, T_3\}$. So, the PDF of T is

$$f_T(t) = 3f(t)[1 - F(t)]^2 = 3\lambda e^{-\lambda t}\left(e^{-\lambda t}\right)^2 = 3\lambda e^{-3\lambda t}, t > 0.$$

This shows that $T \sim Exp(3\lambda)$ and naturally we have $E(T) = 1/3\lambda$.

9. Let X be a continuous random variable with PDF that is symmetric about zero, and let $Y = SX$, where S is a discrete random variable independent of X and has PMF $P(S = 1) = P(S = -1) = 0.5$. Compute $\mathrm{Cov}(X, Y)$. Are X and Y independent? Justify your answer. (10 points)

   Solution: By definition,

$$\mathrm{Cov}(X, Y) = E\big[\big(X - E(X)\big)\big(Y - E(Y)\big)\big] = E\big[\big(X - E(X)\big)\big(SX - E(SX)\big)\big].$$

   Since the PDF of X is symmetric about zero, $E(X) = 0$. In addition, $E(S) = 1 \cdot P(S = 1) + (-1) \cdot P(S = -1) = 0$. Moreover, as X and S are independent, so

$$E(SX) = E(S)E(X) = 0 \cdot 0 = 0.$$

$$\Longrightarrow \mathrm{Cov}(X, Y) = E[X \cdot SX] = E[SX^2] = E(S)E(X^2) = 0.$$

   This shows that X and Y are uncorrelated. However, they are not independent. Consider the joint CDF of (X, Y), by the total law of probability

$$\begin{aligned} F_{X,Y}(x, y) &= P(X \le x, Y \le y) = P(X \le x, SX \le y) \\ &= P(X \le x, X \le y | S = 1)P(S = 1) + P(X \le x, -X \le y | S = -1)P(S = -1) \\ &= \begin{cases} 0.5F_X(\min\{x, y\}) + 0.5[F_X(x) - F_X(-y)], & \text{if } y \ge -x \\ 0.5F_X(\min\{x, y\}), & \text{if } y < -x \end{cases}. \end{aligned}$$

   Since the PDF of X is symmetric about zero, $F_X(-y)$ can also be replaced by $1 - F_X(y)$. As $F_{X,Y}(x, y) \ne F_X(x)F_Y(y)$, X and Y are not independent.

10. A store sells a specific type of goods. The purchase quantity (进货量) per week X and customers' demand on the goods per week Y are independent random variables, both follow $U(10,20)$. Suppose that the profit of the store obtained by selling one unit of goods is 1000 yuan. However, if the demand exceeds the purchase quantity, the store need to order goods from other stores, and the profit per unit of goods is 500 yuan in this case. Compute the expected profit per week of the store by selling this type of goods. (10 points)

    Solution: Let Z denote the profit per week of the store by selling this type of goods. By the description in the problem, $Z = g(X, Y)$ with

$$g(x, y) = \begin{cases} 1000y, & y \le x \\ 1000x + 500(y - x), & y > x \end{cases} = \begin{cases} 1000y, & y \le x \\ 500(x + y), & y > x \end{cases}.$$

    Moreover, by the independence of X and Y, the joint PDF of (X, Y) is

$$f(x, y) = \begin{cases} \dfrac{1}{100}, & 10 < x, y < 20 \\ 0, & \text{otherwise} \end{cases}.$$

    Therefore:

$$E(Z) = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)dxdy$$

$$= \frac{1}{100}\iint_{y \le x, 10 < x, y < 20} 1000ydxdy + \frac{1}{100}\iint_{y > x, 10 < x, y < 20} 500(x+y)dxdy$$

$$= 10\int_{10}^{20}\left(\int_{y}^{20} ydx\right)dy + 5\int_{10}^{20}\left(\int_{10}^{y}(x+y)dx\right)dy = \frac{20000}{3} + 7500$$

≈ 14166.67 (yuan).