STA5002: Mathematical Statistics

Assignment 4 Solution (Dec 26th – Dec 31st)

Note: The solutions only serve as a reference. Some problems may have different methods to reach the same answer.

1. Suppose that $X_1, X_2, \ldots, X_n$ is a sample from $X \sim N(\mu, 1)$, consider the following hypothesis testing problem:

$$H_0:\ \mu = 2\ \ vs.\ H_1: \mu = 3.$$

If the rejection region is $RR = \{\bar{X} \geq 2.5\}$:

(1) Compute the type I and type II error rates if $n = 16$. (10 points)

(2) If we need the type II error rate to be less than 0.01, how large n should be? (5 points)

Solution:

(1) By the definition of type I and type II error, we have:

$$\text{Type I error rate} = P(\boldsymbol{X} \in RR | H_0 \text{ is true}) = P(\bar{X} \geq 2.5 | \mu = 2)$$

$$= P\left(\frac{\bar{X} - 2}{1/\sqrt{n}} \geq \frac{2.5 - 2}{1/\sqrt{n}} | \bar{X} \sim N\left(2, \frac{1}{n}\right)\right) = 1 - \Phi\left(\frac{2.5 - 2}{1/\sqrt{16}}\right) = 1 - \Phi(2)$$

$$= 1 - 0.9772 = 0.0228.$$

$$\text{Type II error rate} = P(X \notin RR | H_1 \text{ is true}) = P(\bar{X} < 2.5 | \mu = 3)$$

$$= P\left(\frac{\bar{X} - 3}{1/\sqrt{n}} < \frac{2.5 - 3}{1/\sqrt{n}} | \bar{X} \sim N\left(3, \frac{1}{n}\right)\right) = \Phi\left(\frac{2.5 - 3}{1/\sqrt{16}}\right) = \Phi(-2)$$

$$= 0.0228.$$

(2) For the type II error rate to be less than 0.01, then:

$$\text{Type II error rate} = P(X \notin RR | H_1 \text{ is true}) = P(\bar{X} < 2.5 | \mu = 3)$$

$$= P\left(\frac{\bar{X} - 3}{1/\sqrt{n}} < \frac{2.5 - 3}{1/\sqrt{n}} | \bar{X} \sim N\left(3, \frac{1}{n}\right)\right) = \Phi\left(\frac{2.5 - 3}{1/\sqrt{n}}\right) = \Phi(-0.5\sqrt{n})$$

$$< 0.01$$

By referring to the normal distribution table:

$$\Longrightarrow -0.5\sqrt{n} < -2.33 \Longrightarrow n \geq (2.33 \times 2)^2 \approx 21.72.$$

Therefore, for the type II error rate to be less than 0.01, we need n to be at least 22.

2. A primary school teacher saw a report in the newspaper: primary school students in

Shenzhen watch 8 hours of TV per week on average. She thinks that the students in her school spend less time watching TV than the reported number. To verify her thought, she randomly surveyed n students in the school, and the average time watching TV per week is obtained as $\bar{X} = 7.5$ (hours), and the corresponding standard deviation is $S = 2$ (hours). Assume that the time of watching TV per week follows a normal distribution, under the two cases below, do you think the teacher is correct? (Use $\alpha = 0.05$)

(1) $n = 16$. (5 points)

(2) $n = 64$. (5 points)

Solution: Let the time of watching TV per week follows $N(\mu, \sigma^2)$, then the hypothesis testing problem is:

$$H_0: \mu = 8 \text{ vs. } H_1: \mu < 8.$$

Since σ^2 is unknown, the one-sample t-test is applied. The test statistic and rejection region are:

$$T = \frac{\sqrt{n}(\bar{X} - 8)}{S}, RR = \{T < t_\alpha(n-1)\}.$$

(1) For $n = 16$, the observed value of T is $t_{\text{obs}} = \frac{\sqrt{16}(7.5-8)}{2} = -1$. By referring to the t-distribution table, $t_{0.05}(15) = -1.753$. Since $t_{\text{obs}} = -1 > -1.753$ which falls out of the rejection region, we fail to reject H_0 at the 0.05 level, i.e., we don't think the teacher is correct.

(2) For $n = 64$, the observe value of T is $t_{\text{obs}} = \sqrt{64}(7.5 - 8)2 = -2$. By referring to the t-distribution table, $-1.671 < t_{0.05}(63) < -1.658$. Since $t_{\text{obs}} = -2 < -1.671$ which falls into the rejection region, H_0 is rejected at the 0.05 level, i.e., we tend to agree with the teacher that the students in her school spend less time watching TV than the reported number.

3. The following table provides the sample observed values of the battery life (in hours) after a charge (一次充电的续航时间) for two types of laptops.

Type 1	5.6	6.3	4.6	5.3	5.0	6.2	5.8	5.1	5.2	5.9	
Type 2	4.0	4.3	4.2	4.0	4.9	4.5	5.2	4.8	4.5	3.9	4.1

Suppose that the two samples come from two independent normal populations.

(1) At the $\alpha = 0.1$ level, are the variances of the battery life for the two types of laptops significantly different? (10 points)

(2) At the $\alpha = 0.01$ level, is the expected battery life of type 1 laptops significantly longer than type 2 laptops? (10 points)

Solution: By the description of the problem, let $X_1, X_2, \dots, X_{10}$ be a sample from population $X \sim N(\mu_1, \sigma_1^2)$, where X denotes the battery life of a type 1 laptop; let $Y_1, Y_2, \dots, Y_{11}$ be a sample from population $Y \sim N(\mu_2, \sigma_2^2)$, where Y denotes the battery life of a type 2 laptop. By calculation, we have:

$$\bar{X} = 5.5, S_1^2 = 0.3044, \bar{Y} = 4.4, S_2^2 = 0.178.$$

(1) The problem is to test $H_0: \sigma_1^2 = \sigma_2^2$ vs. $H_1: \sigma_1^2 \neq \sigma_2^2$. As μ_1 and μ_2 are unknown, the F-test is applied. The test statistic and rejection region are:

$$F = \frac{S_1^2}{S_2^2}, \qquad RR = \left\{F \le F_{\alpha/2}(9,10) \text{ or } F \ge F_{1-\alpha/2}(9,10)\right\}.$$

As $\alpha = 0.1$, from the F-distribution table, $F_{0.95}(9,10) = 3.02$, $F_{0.05}(9,10) = 1/F_{0.95}(10,9) = 1/3.14 \approx 0.3185$. So, $RR = \{F \le 0.3185 \text{ or } F \ge 3.02\}$. The observed value of F is $f_{\text{obs}} = 0.3044/0.178 \approx 1.7104$, which does not fall into the rejection region, so we fail to reject H_0 at the 0.1 level.

(2) The problem is to test $H_0: \mu_1 = \mu_2$ vs. $H_1: \mu_1 > \mu_2$. Since we fail to reject $\sigma_1^2 = \sigma_2^2$ in (1), the two-sample t-test can be applied. The test statistic and rejection region are:

$$T = \frac{\bar{X} - \bar{Y}}{S_\omega \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}, \qquad RR = \{T \ge t_{1-\alpha}(n_1 + n_2 - 2)\} = \{T \ge t_{1-\alpha}(19)\}.$$

As $\alpha = 0.01$, by referring to the t-distribution table, $t_{0.99}(19) = 2.539$, so $RR = \{T \ge 2.539\}$. The pooled variance is

$$S_\omega^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{9 \times 0.3044 + 10 \times 0.178}{19} \approx 0.2379.$$

The observed value of T is

$$t_{\text{obs}} = \frac{5.5 - 4.4}{\sqrt{0.2379 \times \left(\frac{1}{10} + \frac{1}{11}\right)}} \approx 5.1616.$$

Since $t_{\text{obs}} > 2.539$, i.e., it falls into the rejection region, H_0 is rejected at the 0.01 level. Therefore, we agree that the expected battery life of type 1 laptops is significantly longer than type 2 laptops at the 0.01 level.

4. To study the effect of cigarette smoking on platelet aggregation (血小板凝聚), a researcher drew blood samples from 11 individuals before and after they smoked a cigarette and measured the extent to which the blood platelets aggregated. The table below gives the aggregation level:

Before	25	25	27	44	30	67	53	53	52	60	28
After	27	29	37	56	46	82	57	80	61	59	43

Assuming that the difference of aggregation level before and after smoking follows a normal distribution, test whether the aggregation level is significantly higher after smoking. Use $\alpha = 0.01$ and compute the p-value of the test. (10 points)

Solution: This is a paired comparison problem. Let D_i be the difference of aggregation level before and after smoking for the ith individual (after - before), then:

Before	25	25	27	44	30	67	53	53	52	60	28
After	27	29	37	56	46	82	57	80	61	59	43
D_i	2	4	10	12	16	15	4	27	9	-1	15

Assuming that $D_i \sim N(\mu, \sigma^2)$, the hypothesis testing problem is:

$$H_0: \mu = 0 \text{ vs. } H_1: \mu > 0.$$

As σ^2 is unknown, the one-sample t-test would be applied. The test statistic is:

$$T = \frac{\sqrt{n}(\bar{D} - 0)}{S_D} \Longrightarrow t_{\text{obs}} = \frac{\sqrt{11} \times 10.2727}{7.9761} \approx 4.2716.$$

So, the p-value is $P(T \geq 4.2716)$ where $T \sim t(10)$ under H_0. By referring to the t-distribution table, we have

$$0.0005 < \text{p} - \text{value} < 0.001 \Longrightarrow \text{p} - \text{value} < \alpha = 0.01.$$

Therefore, H_0 is rejected at the 0.01 level, i.e., we agree that the aggregation level is significantly higher after smoking at the 0.01 level.

5. Time magazine reported the result of a telephone poll of 800 adult Americans. The question asked was: "Should the federal tax on cigarettes be raised to pay for health care reform?". The results of the survey were: 351 out of 605 non-smokers answered "yes" and 41 out of 195 smokers answered "yes". At the $\alpha = 0.05$ level, can we conclude that the two populations (non-smokers and smokers) differ significantly with respect to their opinions? (10 points)

Solution: Let p_1 and p_2 be the proportion who reply "yes" of the non-smoker and smoker populations, respectively. Then the hypothesis testing problem is:

$$H_0: p_1 - p_2 = 0 \text{ vs. } H_1: p_1 - p_2 \neq 0.$$

Under H_0, $p_1 = p_2 = p$, the estimated value of p is

$$\hat{p} = \frac{351 + 41}{800} = 0.49.$$

The test statistic is

$$U = \frac{\bar{X} - \bar{Y}}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \Longrightarrow u_{\text{obs}} = \frac{\frac{351}{605} - \frac{41}{195}}{\sqrt{0.49 \times 0.51 \times \left(\frac{1}{605} + \frac{1}{195}\right)}} \approx 8.9859.$$

The rejection region is $RR = \{|U| \geq u_{1-\alpha/2}\} = \{|U| \geq u_{0.975}\} = \{|U| \geq 1.96\}$. Since u_{obs} falls into the rejection region, H_0 is rejected at the 0.05 level. There is enough evidence at the 0.05 level to conclude that the two populations differ with respect to their opinions concerning imposing a federal tax to help pay for health care reform.

6. Suppose that $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ is a sample from population $X \sim N(\mu, \sigma^2)$ where σ^2 is known. Obtain the rejection region of the likelihood ratio test for the testing problem $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$. (10 points)

Solution: The joint PDF of the sample is

$$f(\boldsymbol{x}; \mu) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}.$$

The two parameter spaces are:

$$\Theta_0 = \{\mu: \mu = \mu_0\},\ \Theta = \{\mu: -\infty < \mu < \infty\}.$$

It is not difficult to obtain that the MLE of μ in Θ is

$$\hat{\mu} = \bar{X} \Longrightarrow \sup_{\theta \in \Theta} f(\boldsymbol{x}; \mu) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right\}.$$

The MLE of μ in Θ_0 is simply

$$\hat{\mu} = \mu_0 \Longrightarrow \sup_{\theta \in \Theta_0} f(\boldsymbol{x}; \mu) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_0)^2\right\}.$$

Since

$$\sum_{i=1}^{n}(x_i - \mu_0)^2 = \sum_{i=1}^{n}(x_i - \bar{x} + \bar{x} - \mu_0)^2 = \sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2,$$

then the likelihood ratio is

$$\Lambda(\boldsymbol{x}) = \frac{\sup_{\theta \in \Theta} f(\boldsymbol{x}; \mu)}{\sup_{\theta \in \Theta_0} f(\boldsymbol{x}; \mu)} = \exp\left\{\frac{n(\bar{x} - \mu_0)^2}{2\sigma^2}\right\} = \exp\left\{\frac{u^2}{2}\right\},$$

where u corresponds to statistic $U = \sqrt{n}(\bar{X} - \mu_0)/\sigma$. To determine the rejection region

$RR = \{\Lambda \geq c\}$, consider the type I error rate:

$$P(\Lambda \geq c|H_0) = P\left(\exp\left\{\frac{U^2}{2}\right\} \geq c \,\middle|\, H_0\right) = P\left(|U| \geq \sqrt{2\log c}\,|H_0\right) = \alpha$$

$$\Longrightarrow \sqrt{2\log c} = u_{1-\alpha/2} \Longrightarrow c = \exp\left\{\frac{u_{1-\alpha/2}^2}{2}\right\}.$$

Therefore, the rejection region of the likelihood ratio test is $RR = \left\{\Lambda \geq \exp\left(\frac{u_{1-\alpha/2}^2}{2}\right)\right\}$, or equivalently $RR = \left\{|U| \geq u_{1-\alpha/2}\right\}$.

7. In a one-way analysis of variance problem, factor A has four levels, each level was assigned five individuals. Please complete the following ANOVA table (10 points)

ANOVA Table					
Source	DF	SS	MS	F-value	P-value
Factor A		37.8			
Error					
Total		69.8			

Solution: The completed ANOVA table is given below

ANOVA Table					
Source	DF	SS	MS	F-value	P-value
Factor A	3	37.8	12.6	6.30	0.005
Error	16	32.0	2.0		
Total	19	69.8			

8. A food company designed three new packaging (外包装) for a product of potato chips. To see which packaging is most popular, 16 grocery stores (食品杂货店) with similar size and location were selected to perform an experiment, 5 stores sold products with the first packaging, another 5 stores sold products with the second packaging, and the remaining 6 stores sold products with the third packaging. The sales data after one month are provided below:

Packaging 1	12	18	14	15	16	
Packaging 2	14	12	13	12	14	
Packaging 3	19	17	21	24	18	21

Assume that all the variances are equal. Apply the one-way ANOVA to see if the three packaging have the same effect on product sale. Please complete the problems by **hand calculation** and use $\alpha = 0.05$. (15 points)

Solution: With the data provided, the sample means and variances of the three groups are:

$$\bar{y}_1 = \frac{12 + 18 + \cdots + 16}{5} = 15, \bar{y}_2 = \frac{14 + 12 + \cdots + 14}{5} = 13, \bar{y}_3 = \frac{19 + 17 + \cdots 21}{6} = 20.$$

$$s_1^2 = \frac{(12 - 15)^2 + \cdots + (16 - 15)^2}{4} = 5, s_2^2 = \frac{(14 - 13)^2 + \cdots + (14 - 13)^2}{4} = 1,$$

$$s_3^2 = \frac{(19 - 20)^2 + \cdots + (21 - 20)^2}{5} = 6.4.$$

The overall sample mean and pooled sample variance are:

$$\bar{y} = \frac{12 + 18 + \cdots + 21}{16} = 16.25, s^2 = \frac{\sum_{i=1}^{3}(n_i - 1)s_i^2}{16 - 3} = \frac{4 \times 5 + 4 \times 1 + 5 \times 6.4}{13} \approx 4.308.$$

To perform the one-way ANOVA, compute the observed value of SSA and SSE:

$$SSA = n_1(\bar{y}_1 - \bar{y})^2 + n_2(\bar{y}_2 - \bar{y})^2 + n_3(\bar{y}_3 - \bar{y})^2 = 145.$$

$$SSE = (n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + (n_3 - 1)s_3^2 = 56.$$

Therefore, we could construct the ANOVA table:

ANOVA Table					
Source	DF	SS	MS	F-value	P-value
Packaging	2	145	72.50	16.83	<0.001
Error	13	56	4.31		
Total	15	201			

As the p-value $< 0.001 < \alpha = 0.05$, $H_0: \mu_1 = \mu_2 = \mu_3$ is rejected at the 0.05 level, i.e., the three packaging do not all have the same effect on product sale.