STA5002: Mathematical Statistics

Assignment 1 Solution (Sep 29th – Oct 9th)

Note: The solutions only serve as a reference. Some problems may have different methods to reach the same answer.

1. For two events A and B from the same sample space Ω, if $P(A) = 1/4$ and $P(B^c) = 1/5$, can A and B be disjoint? Please provide your explanation. (10 points)

   Solution: No. If A and B are disjoint, then A must be a subset of B^C, so that $P(A) \leq P(B^C)$. However, $P(A) = 1/4 > P(B^C) = 1/5$. Therefore, A and B cannot be disjoint.

2. Three babies are given a weekly health check at a clinic, and then returned randomly to their mothers. What is the probability that at least one baby goes to the right mother? (10 points)

   Solution: Let E_i be the event that baby i is reunited with its mother. We need $P(E_1 \cup E_2 \cup E_3)$, where we can use the result

   $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$

   For any $A, B, C,$ the individual probabilities are $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$. The pairwise joint probabilities are equal to $\frac{1}{6}$, since $P(E_1 E_2) = P(E_2 | E_1) P(E_1) = (\frac{1}{2})(\frac{1}{3})$, and the triplet $P(E_1 E_2 E_3) = \frac{1}{6}$ similarly. Hence our final answer is

   $$\frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{6} = \frac{2}{3}.$$

3. I have in my pocket ten coins. Nine of them are ordinary coins with equal chances of coming up head and tail when tossed and the tenth has two heads.

   (a) If I randomly take a coin from my pocket, toss the coin and it comes up head, what is the probability that it is the coin with two heads? (10 points)

   (b) If I toss the coin one further time and it comes up head again, what is the probability that it is the coin with two heads? (10 points)

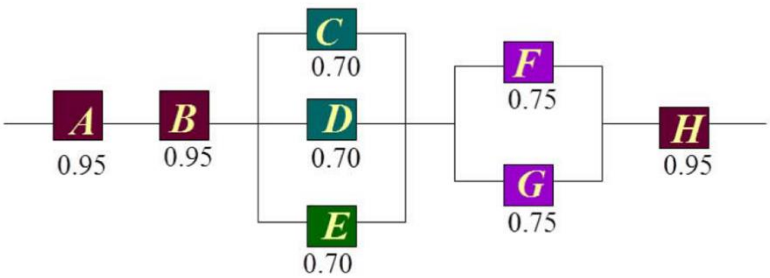
   Solution:

   (a) Denote by H the event that we get a head when tossing the coin and by D the event that the coin is the one with two heads. Then $P(H|D) = 1$ and $P(D) = \frac{1}{10}$ and the probability that we are interested is

$$P(D|H) = \frac{P(H|D)P(D)}{P(H|D)P(D) + P(H|D^C)P(D^C)} = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{1}{2} \times \frac{9}{10}} = \frac{2}{11}.$$

(b) We use the equation in (a) again, but replace $P(D)$ by $\frac{2}{11}$. Then the probability that a coin has two heads if we get two heads by tossing it twice is $\frac{\frac{2}{11}}{\frac{2}{11}+\frac{1}{2}\times\frac{9}{11}} = \frac{4}{13}$.

4. In the following circuit, A, B, C, D, E, F, G, H are independent components. The values labeled under the components are their corresponding probabilities of working properly. What is the probability for the whole circuit to work properly? (10 points)



Solution: Let $W = \{\text{the circuit works normally}\}$. Then

$$P(W) = P(AB)P(C \cup D \cup E)P(F \cup G)P(H).$$

Consider

$$P(C \cup D \cup E) = 1 - P(C^C)P(D^C)P(E^C) = 1 - 0.3^3 = 0.973,$$
$$P(F \cup G) = 1 - P(F^C)P(G^C) = 1 - 0.25^2 = 0.9375,$$

therefore:

$$P(W) = 0.95 * 0.95 * 0.973 * 0.9375 * 0.95 = 0.782.$$

5. A bus has 20 passengers on board. There are 10 bus stops in total along the way. The bus only stops at a bus stop if at least one passenger needs to get off the bus. Let X denote the number of stops that the bus would stop, find $E(X)$ (assume that the probability of each passenger to get off the bus at any stop is the same, and passengers are independent). (10 points)

Solution: Let

$$X_i = \begin{cases} 1, & \text{at least one passenger get off the bus at the } i\text{th stop} \\ 0, & \text{no passenger gets off the bus at the } i\text{th stop} \end{cases}, \quad i = 1,2,\dots,10$$

$X_i = 0$ suggest that no passenger gets off the bus at the ith stop. Therefore:

$$P(X_i = 0) = \left(\frac{9}{10}\right)^{20}, P(X_i = 1) = 1 - \left(\frac{9}{10}\right)^{20}, i = 1, 2, \dots, 10.$$

Since $X = X_1 + X_2 + \cdots + X_{10}$, so:

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_{10}) = 10E(X_1) = 10\left[1 - \left(\frac{9}{10}\right)^{20}\right] = 8.784.$$

6. Phone calls are received at a certain residence as a Poisson process with $\lambda = 2$ per hour.
   (a) If Amber takes a 10-minute shower, what is the probability that the phone rings during that time? (5 points)
   (b) How long can her shower be if she wishes the probability of receiving no phone calls to be at most 0.5? (5 points)

   Solution:

   (a) Let X be the number of calls in $[0, t)$, then $X \sim P(2t)$. For $t = \frac{10}{60} = 1/6$ (in hours), $X \sim P(1/3)$. The probability that the phone rings during the shower is:

   $$P(X > 0) = 1 - P(X = 0) = 1 - e^{-\frac{1}{3}} = 0.283.$$

   (b) For $X \sim P(2t)$, the probability of receiving no phone calls to be at most 0.5 means that $P(X = 0) \leq 0.5$. So

   $$P(X = 0) = e^{-2t} \leq 0.5 \implies t \geq \frac{\ln(2)}{2} = 0.347(\text{hours}) \approx 21(\text{minutes}).$$

7. Suppose that the salaries of employees in a company is approximately normally distributed. Suppose you know that 33% employees earn less than \$66,000 per year and 33% employees earn more than \$88,000 per year. What is the probability that an employee makes between \$82,000 and \$92,000 per year? (10 points)

   Solution: Let the salary of an employee in the company be $X \sim N(\mu, \sigma^2)$. From the description, we have:

   $$P(X < 66000) = 0.33, P(X > 88000) = 0.33.$$

   By looking at the standard normal distribution, we obtain that:

   $$P(X < 66000) = P\left(\frac{X - \mu}{\sigma} < \frac{66000 - \mu}{\sigma}\right) = 0.33 \implies \frac{66000 - \mu}{\sigma} = -0.44$$

$$P(X > 88000) = P\left(\frac{X-\mu}{\sigma} > \frac{88000-\mu}{\sigma}\right) = 0.33 \Longrightarrow \frac{88000-\mu}{\sigma} = 0.44$$

Therefore, $\mu = 77000$ and $\sigma = 25000$. Then

$$P(82000 < X < 92000) = P\left(\frac{82000-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{92000-\mu}{\sigma}\right) = P(0.2 < Z < 0.6)$$
$$= 0.7257 - 0.5793 = 0.1464 = 14.64\%.$$

8. The CDF of the Weibull distribution is ($\alpha > 0,\ \beta > 0$)

$$F(x) = 1 - e^{-(x/\alpha)^\beta}, x \geq 0.$$

(a) If W follows a Weibull distribution, find the PDF of $X = (W/\alpha)^\beta$. (5 points)

(b) How could Weibull random variables be generated from a Uniform random number generator? (5 points)

Solution:

(a) Consider

$$F_X(x) = P(X \leq x) = P\big((W/\alpha)^\beta \leq x\big) = P\big(W \leq \alpha x^{1/\beta}\big) = 1 - e^{-x}$$

and the PDF of X is

$$f_X(x) = F_X'(x) = e^{-x}, x \geq 0.$$

(b) Since $F(x) = 1 - e^{-(x/\alpha)^\beta}$, then $F^{-1}(y) = \alpha[-\ln(1-y)]^{1/\beta}$ $(0 < y < 1)$. So, to generate a sample from the Weibull distribution:

Step 1: generate a sample $y_1, y_2, \dots, y_n$ from $U(0,1)$.

Step 2: compute $x_i = F^{-1}(y) = \alpha[-\ln(1-y_i)]^{1/\beta}$, then $x_1, x_2, \dots, x_n$ is a sample from the Weibull distribution.

9. Let X be a random variable with the moment generating function

$$m_X(t) = \frac{1}{1-5t}, t < \frac{1}{5}.$$

Find $E(X+5)^2$. (10 points)

Solution:

$$E(X+5)^2 = E[X^2 + 10X + 25] = E[X^2] + 10E(X) + 25.$$

Take the first and second derivative of the moment generating functions $m_X(t)$:

$$m_X'(t) = \frac{5}{(1-5t)^2}, m_X''(t) = \frac{50}{(1-5t)^3}$$

$$\Longrightarrow E(X) = m_X'(0) = 5, E(X^2) = m_X''(0) = 50$$

$$\Longrightarrow E(X+5)^2 = 50 + 10 \times 5 + 25 = 125.$$